Digital Currency Runs

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Abstract

Digital currency is designed to compete with central bank fiat money and the banking system but may create new financial stability risk. Central banks are considering issuing their own fiat public digital currency in response. This paper shows that privately issued digital currency, such as bitcoin, may be adopted in reaction to distortionary central bank inflation on fiat money. Banks that take private digital currency deposits can emerge to provide efficient liquidity risk sharing without the inflationary risk of fiat money. Rather than displacing banks, private and public digital currency threaten a new form of banking crises caused by disintermediation runs through withdrawals of digital currency. A central bank can act as lender of last resort to prevent the threat of such digital currency runs for banks with public but not private digital currency deposits. There is a trade-off for private digital currency that avoids the costs of central bank inflation but is subject to fragility through digital currency runs.

Keywords: Digital currency, fiat money, bank runs, lender of last resort
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1 Introduction

The rapid development of digital currency has prompted widely acclaimed interest in its potential impact on the financial system and the economy. A primary motivation behind the development of digital currency, such as the advent of bitcoin by Nakamoto (2008), is that it has a fixed supply rather than the discretionary supply of central bank fiat money. A second key motivation is that digital currency allows for payments without need for banks. Questions have emerged about whether digital currency might eventually displace fiat money and the banking system. In response, central banks worldwide are considering, and some have begun, issuing their own form of digital currency.[2] Concerns have also arisen about whether digital currency might create fragility in the financial system.[3]

This paper develops a model of digital currency introduced into an economy with banks and a central bank. Privately issued digital currency, such as bitcoin, may be adopted if the central bank creates distortionary inflation on fiat money, regardless of whether the central bank tries to compete with its own publicly issued digital currency. While digital currency can replace banks for providing a payments, banks that take digital currency deposits can emerge to provide efficient liquidity risk sharing without the inflationary risk of fiat money. Rather than displacing banks, digital currency threatens a new form of banking crises caused by digital currency runs on banks that take digital currency deposits. A central bank can act as lender of last resort to prevent the threat of such withdrawal runs for banks taking publicly issued but not privately digital currency deposits.

An economy with a monetary system that is based on a private digital currency instead of central bank fiat money is a viable possibility, as suggested by Raskin and Yermack (2016). Bitcoin has had increasing adoption in several countries with high inflation problems including Venezuela, Iran, Argentina, Ukraine, Zimbabwe, and other African countries.[4]

Raskin and Yermack (2016) also point out widespread expectations that digital

[2] See Bech and Rodney (2017), Bordo and Levin (2017), Broadbent (2016), Cawrey (2014), Jack and Suri (2011), Popper (2015), and Rosenfeld (2015).

[3] For example, see Winkler (2015) and Nelson (2017).

[4] For example, see Raskin, Max (2012) and Urban (2017).

currency will disintermediate banks by ending fractional reserve banking.[5] The establishment of either a public or private digital currency in an economy could allow for holding money and an efficient digital means of payment without use of the banking system.

However, fractional reserve banking based on paying a return on deposits and making loans denominated in bitcoin is emerging. Mastercard has recently won patents, and is applying for additional ones, for methods and systems for a fractional reserve digital currency bank.[6] Several platforms already provide bitcoin savings accounts that pay interest generated by returns from lending bitcoin for leveraged trading.[7] In addition, empirical evidence demonstrates that despite the ability for the growing fintech economy to operate outside of financial intermediation, banking in effect reemerges.[8] While bitcoin-based banks are showing signs of an initial emergence, broad concerns that private digital currency might create systemic financial fragility have as yet to be formalized and studied.[9]

In my model, two forms of digital currency are introduced into an economy based on banking with fiat money. Digital currency that is issued by the central bank, referred to as *public digital currency*, is fiat money that can be held by consumers in direct accounts at the central bank and hence outside of the banking system.[10] Digital currency such as bitcoin that is privately issued, referred to as *private digital currency*, can also be held as a form of outside money directly by consumers. Both forms of digital currency can be used by consumers to make payments for buying and selling goods without relying on holding bank deposits to make payments for such transactions.

[5] For example, see Vigna and Casey (2015).

[6] See Madore (2018), Suberg (2018), and U.S. Patent and Trademark Office (2018).

[7] See See Kharif (2019), Lielacher (2017), Tomasicchio (2016), Cruz (2015), Johnson (2015), and for a list of such platforms, https://cryptalker.com/bitcoin-interest and https://cryptalker.com/bitcoin-savings.

[8] Balyuk and Davydenko (2018) show that fintech platforms designed for direct peer-to-peer lending are evolving toward becoming essentially online intermediaries in the form of banks that take investment from passive lenders and make active investment decisions for lending to borrowers.

[9] In blog posts, Posner (2015a,b) is one of the few to point out that if bitcoin becomes widely adopted, fractional reserve banking denominated in bitcoin will be a natural outcome because of the value that banks provide, but that financial crises in a bitcoin-based banking system will also occur.

[10] Raskin and Yermack (2016) describe how a central bank digital currency would enable households to hold such public digital currency directly in accounts at the central bank instead of in deposit accounts at commercial banks.

I show that if the central bank has a bias for short-term output, there is distortionary fiat inflation that can lead to privately issued digital currency being adopted and displacing fiat money. Banks switch to taking private digital currency deposits to provide consumers with efficient liquidity risk sharing without the inflationary risk of fiat money. The economy based on fiat money can transition to a private digital currency while still featuring a fractional reserve banking system similar to that with fiat money.

However, digital currency runs arise as a new threat on banks that take deposits in either privately or publicly issued digital currency. Such runs can disintermediate banks and cause their failure based on the ability of depositors to withdraw and hold digital currency as a store of value and means of payment outside of the banking system. The central bank can prevent the threat of these runs for banks with public digital currency deposits but not for banks with private digital currency deposits. The discretion that a central bank has over public digital currency, as with traditional fiat reserves, allows for distortionary inflation but also for acting as lender of last resort to prevent runs on banks with public digital currency deposits. There is a trade-off for holding private digital currency. It avoids the costs of central bank inflation borne by fiat money but loses the liquidity value creation of bank deposits if held directly. If instead, it is held as bank deposits, it is subject to fragility in the form of digital currency runs.

Indeed, the Federal Reserve was originally created for the primary purpose of being able to provide an "elastic supply of currency" in order to act as a lender of last resort. But, the Fed's discretion to increase the money supply has often come under pressure since the founding of the Fed. The earliest call for a privately issued digital currency to constrain the elastic supply of fiat money is likely by Milton Friedman. In 1999, Friedman famously foresaw and welcomed the opportunities for an internet-based digital currency to be supplied inelastically in an algorithmic manner according to an automated rule to constrain monetary policy discretion, as described by Raskin and Yermack (2016).

Friedman's foresight reflects two key features of private digital currency. First, it is supplied based on algorithmic quantities, whereas central banks have discretion over the supply of fiat money. In my model, I assume that private digital currency is supplied with a fixed quantity. Whereas, the central bank does not have the ability

to commit to a fixed supply of fiat reserves or fiat public digital currency.

Second, private digital currency utilizes a decentralized distributed ledger with blockchain technology and requires a protocol to achieve consensus for payments transactions in such a 'trustless' environment.[11] Public digital currency payments under current consideration would likely utilize a 'trusted' centralized central bank ledger. With developments in methods for private digital currency payments to support consensus for transactions in a more cost effective manner, such as with proof-of-stake rather than proof-of-work protocols,[12] or with second-layer protocols such as the Lightning Network to increase scalability,[13] private digital currency has the potential to be used as an efficient means of digital payments similar to or even more advanced than electronic payments that are cleared and settled within the banking system. For this reason, I make the simplifying assumption in the paper of no transactions costs for payments in the economy and financial system made by using either digital currency as outside money or bank deposits as inside money.[14]

The model provides several additional novel insights. I show the resiliency of the banking system based on an elastic price level that arises in a parsimonious model of the economy with fiat money, which allows for a nominal unit of account, in contrast to models of real economies. As unit of account, fiat money or private digital currency allows for elastic prices and a flexible real value of nominal bank deposits that provides

[11] Kroll et al. (2013) examine bitcoin as a consensus game using costly computational mining as proof-of-work for transaction consensus, and which also requires a separate governing consensus for the rules of the bitcoin protocol. Biais et al. (2018a) show that bitcoin transaction consensus using the mining proof-of-work protocol is a Markov perfect equilibrium but that consensus over the protocol is a coordination game with multiple equilibria. Cong et al. (2018) examine methods for moderating the natural concentration of mining pools, and Easley et al. (2017) explain market-based transaction fees charged in addition to mining rewards.

[12] Saleh (2018a,b) shows that protocols such as proof-of-stake or proof-of-burn can overcome the large computing resources costs required for proof-of-work consensus protocols, such as for bitcoin, which Parham (2017) demonstrates are prohibitive on a large scale.

[13] Poon and Dryja (2016) describe how the Lightning Network, which has reached increasing success in recent small-value tests, acts as a decentralized network off of the bitcoin blockchain for micropayments in bitcoin, with net payments then transacted on the bitcoin blockchain.

[14] Payments Canada et al. (2018) demonstrate the potential for widespread banking payments without reliance on a central bank that would be required for banks to take private digital currency deposits. They describe the development and testing in Canada for efficiently settling large-scale wholesale interbank payments with distributed ledger technology. A "notary node" consensus model shows promise for settlement finality, which is required but not achieved with a proof-of-work protocol. Parlour et al. (2017) show that fintech innovation in the bank payment system can reduce banks' need for intermediate liquidity in the interbank market, which results in an increase in bank lending and productive efficiency.

optimal risk sharing against asset and liquidity risk in a general equilibrium setting. Even when banks are not required for an efficient payment system, banking still occurs because of the benefits of maturity and risk transformation of illiquid assets for the efficient provision of liquidity to the economy. Private digital currency can act as a form of outside money that banks hold in the form of private reserves, similar to the case of fiat money as reserves, to enable standard fractional reserve banking.

Yet, while both public and private digital currency can have an elastic real value, a primary distinction between a private and public digital currency is that a central bank can provide an elastic supply of public digital currency, as with fiat money, but not privately issued digital currency. Private digital currency does not act as a threat to merely discipline the central bank to lower fiat inflation, because the central bank faces a time-inconsistency problem. The central bank would not be credible if it tried to commit to lower inflation. The central bank also cannot constrain itself from creating fiat inflation by issuing public digital currency. In contrast to inflationary fiat money, private digital currency requires a deflationary price level to provide a return for holding it as a store of value.

Digital currency has been recently studied, along with blockchain technology utilized with distributed decentralized ledgers more broadly, in the rapidly growing finance and economics literature on fintech.[15] Current papers on digital currency, banking, and central bank policy highlight several potential benefits and costs of private and public digital currency. These papers focus on private digital currency competing against monopolist central bank money,[16] public digital currency competing against bank deposits,[17] and competition among private digital currencies,[18] but

[15] Raskin and Yermack (2016) highlight that debates over private digital currency as competition to fiat money is demanding a resurgence in classical monetary economic theory based on von Mises (1912), Hayek (1976) and Mundell (1998).

[16] Abadi and Brunnermeier (2018) show that because of free entry and distributed ledger fork competition, private digital currencies do not produce profits for the issuer or miners but provide competition that only partially constrains central bank profits arising through monopoly power as a centralized intermediary of fiat money and payments. Schilling and Uhlig (2018) show that for a central bank with commitment to maintain the real value of fiat money, there is exchange-rate indeterminacy for the price of private digital currency.

[17] Andolfatto (2018) finds that interest-bearing central bank digital currency constrains the profit but does not disintermediate monopolistic banks and may even lead to their expansion by providing competition for banks to increase deposit rates. In contrast, Keister and Sanches (2018) find that central bank digital currency increases exchange efficiency in a search economy but crowds out investment by banks that rely on real deposits.

[18] Fernandez-Villaverde and Sanches (2017a,b) find that competition among private digital curren-

they do not examine financial stability concerns.

In order to focus on the risks of digital currency runs, I shut down other channels affecting digital currency as money that have been studied. For example, the potential for private digital currency to be widely adopted as money is viewed in part as an economic coordination problem. Bitcoin and other private digital currencies have displayed extreme price volatility, which limits their acceptance and use. However, several studies argue that the increasing acceptance and use of private digital currency will lead to a more stable value, further supporting its use.[19] Several papers also tie the extreme volatility of bitcoin to the proof-of-work protocol,[20] which may be fundamentally overcome through alternative protocols, as demonstrated by Saleh (2018b).

Additionally, whether private digital currency can displace central bank fiat money is in part a political and technological question. While bitcoin gained early notoriety for use in black markets, recent evidence demonstrates that such illicit use is diminishing because of its public blockchain transaction history, which allows authorities to track down illegal users.[21] Meanwhile, bitcoin is increasingly being used for legitimate transactions,[22] in financial markets such as CME bitcoin futures, and in financial services provided by major banks such as Goldman Sachs. Several less developed countries have struggled between the extremes of officially supporting the adoption of bitcoin and banning its use.[23] However, the development of broader applications of blockchain technology beyond digital currency may become so widespread

cies may implement efficient allocations in a search economy with productive capital but otherwise require unconventional methods for central bank monetary policy.

[19] Bolt and van Oordt (2016) show how the price volatility of private digital currency is driven by speculators but decreases as it becomes more widely adopted by consumers and accepted by merchants. Cong et al. (2018) explain the volatility of private digital currency based on the feedback-loop dynamics of it being adopted for transactions. Li and Mann (2018) point to initial coin offerings (ICOs) for investment in private digital currency platforms that can solve the adoption coordination problem. Sockin and Xiong (2018) show that the price and volume of private digital currency transactions act as coordination devices that determine whether there is high, low, or no transactions with the digital currency. Kim (2015) uses empirical evidence based on pre-blockchain based virtual currencies to argue that bitcoin volatility driven by speculators will significantly decrease over time.

[20] Biais et al. (2018b) provide an OLG model and empirical evidence that costly mining determines bitcoin's fundamental value based on the net present value of transactional benefits but also drives large volatility. Pagnotta and Buraschi (2018) show that mining costs being paid in bitcoin amplifies the impact of supply and demand shocks on its price volatility.

[21] See Jawaheri et al. (2018), Meiklejohn (2016), and Bohannon (2016).

[22] See Tasca et al. (2018).

[23] See Hosain (2018).

and ubiquitous in the financial system and economy that digital currency may be a required complementary feature.[24]

The paper proceeds with the general model of the economy with banking, fiat money, and digital currency introduced in Section 2. Section 3 presents the baseline equilibrium with fiat reserves, followed by the analysis of public and private digital currency in section 4. Digital currency runs are studied in section 5, and section 6 provides concluding remarks. All proofs are contained in the appendix.

2 Model

The model builds on the theory of nominal bank contracts as developed in Allen and Skeie (2018), Allen et al. (2014), and Skeie (2004, 2008). They show how nominal bank contracts with fiat central bank money, and without consideration of a short-term central bank bias, can provide depositors with optimal consumption and financial stability against liquidity and asset risk.[25] The model also builds on the provision of liquidity provided by banks that enable runs (Diamond and Dybvig, 1983)[26] and relates to the theory of banking liquidity and fragility in the context of interbank lending;[27] the role of lending money between banks, central bank lending and injections of money, and demand deposits paid in money in models of real bank de-

[24] Applications of blockchain include more efficient smart financial contracting (Cong and He, 2018), managing trading transparency in financial markets to increase investor welfare (Malinova and Park, 2017), and the market for, and regulation of, financial reporting and auditing (Cao et al., 2018).

[25] Conditions for bank runs and contagion with nominal bank contracts are shown by Skeie (2004) as arising from interbank market liquidity freezes, and by Diamond and Rajan (2006) and Champ et al. (1996) as arising to due to withdrawals of currency out of the banking system based on consumer purchases of goods that must be made with traditional paper currency. Diamond and Rajan (2006) further show that nominal contracts do not protect from bank runs caused by heterogeneous shocks in asset returns.

[26] Bank liquidity creation and fragility is further developed based on the insensitivity to information of deposits (Gorton and Pennachi, 1990, and Dang et al., 2013), contracts relative to markets (Allen and Gale, 2004), global games informational structures (Goldstein and Pauzner, 2005), central bank interest rate policy (Freixas et al., 2011), efficient risk management (DeAngelo and Stulz, 2015), bailout policy (Shapiro and Skeie, 2015), and central bank balance sheet policy (Martin et al., 2016, 2018). Dynamic models of bank runs include Brunnermeier and Sannikov (2016), Gertler and Kiyotaki (2015), Martin et al. (2014a, 2014b), Brunnermeier and Oehmke (2013), and He and Xiong (2012).

[27] Bhattacharya and Gale (1987), Bhattacharya and Fulghieri (1994), Allen and Gale (2000a), Diamond and Rajan (2005), Acharya and Skeie (2011), and Ashcraft et al. (2011).

posits;[28] liquidity runs and bank insolvency tied to bank lending contracts;[29] systemic risk triggered from idiosyncratic bank losses;[30] and interbank payments and lending operating through clearinghouse systems for transferring and settling payments between banks.[31]

2.1 Real economy

The model has an infinite number of *dates* $\tau = 0, 1, ..., \infty$. Within each date τ, there are three *periods*, $t \in \{0, 1, 2\}$. At each date τ, there is a new generation of consumers born at period $t = 0$ and live for one or two periods. Consumers are ex-ante identical and are endowed with $e_{0,\tau} = 1$ goods per unit mass of the new generation. There is also free entry of competitive, risk neutral banks and firms who have no endowment and are infinitely-lived.

Within date τ, firms store an amount of goods $g_{t,\tau}$ at period $t \in \{0, 1\}$ for safe, short-term liquidity for a return of one at $t + 1$. However, goods cannot be stored between dates; i.e., goods cannot be stored at period $t = 2$. Firms invest an amount of goods $a_{0,\tau}$ at period $t = 0$ in the form of risky, long-term illiquid assets. An amount $a_{1,\tau} \leq a_{0,\tau}$ of these assets are liquidated at $t = 1$ for a salvage return of $r_1 \in (0, 1)$ at $t = 1$, where r_1 is a constant. The remaining assets $a_{0,\tau} - a_{1,\tau}$ that are not liquidated have a random return $r_{2,\tau} \in (r_1, r_2^{\max})$ at $t = 2$ with expected return $E[r_{2,\tau}] = \bar{r}_2 > 1$.

At each date τ, a random fraction $\lambda_\tau \in (0, 1)$ of consumers have a privately observed liquidity shock and need to consume at $t = 1$, where $E[\lambda_\tau] = \bar{\lambda}$. These 'early consumers' have utility given by $U = u(c_{1,\tau})$. The remaining fraction $1 - \lambda_\tau$ of consumers do not receive a liquidity shock. These 'late consumers' are indexed by $i \in I \equiv [\lambda, 1]$ and have utility $U = u(c^i_{1,\tau} + c^i_{2,\tau})$, where $c^i_{t,\tau}$ is consumption at $t \in \{1, 2\}$. The utility function $u(\cdot)$ is assumed to be twice continuously differentiable, strictly concave, satisfy Inada conditions $u'(0) = \infty$ and $u'(\infty) = 0$, and have a coefficient of relative risk aversion $\frac{-cu''(c)}{u'(c)} > 1$.

The new generation of consumers at each date τ has a mass size n^τ, with $n = \bar{r}_2$,

[28] Allen and Gale (1998), Gale and Vives (2002), Freixas et al. (2000, 2003), Freixas and Holthausen (2001), Rochet and Vives (2004), and Chang and Velasco (2000).

[29] Holmström and Tirole (1998) and Diamond and Rajan (2005).

[30] Rochet and Tirole (1996) and Aghion et al. (2000).

[31] Furfine (1999), Henckel (1999), Flannery (1996) and Hancock and Wilcox (1996).

which implies that the aggregate mass of endowment goods at period $t = 0$ of date τ is $n^\tau e_{0,\tau} = n^\tau = \bar{r}_2^\tau$. The aggregate state $(\lambda_\tau, r_{2,\tau})$ is realized and observable but not verifiable at $t = 1$ of date τ.

2.2 Fiat money and digital currency

Fiat reserves At period $t = 0$ of date $\tau = 0$, the central bank issues to banks $\bar{M} \geq 0$ of fiat reserves, which is fiat outside money that only banks can hold in accounts at the central bank.

Digital currency There are two forms of digital currency considered. The first type is public digital currency issued by the central bank. Public digital currency is fiat outside money and is equivalent to fiat reserves with the exception that it can also be held by consumers and firms in the form of accounts at the central bank and outside of the banking system. Public digital currency held by banks is equivalent to fiat reserves.

The second type of digital currency is private digital currency issued by a private issuer, such as bitcoin. Consumers and firms can hold private digital currency outside of the banking system, while banks can also hold private digital currency as a form of private reserves.

Specifically, at date $\tau' > 0$, an amount $\bar{M}^\iota \geq 0$ indexed by $\iota \in \{v, s\}$ of digital currency is received by the new generation of consumers at period $t = 0$ as outside money, where $\bar{M}^v$ is private digital currency (e.g., bitcoin), and $\bar{M}^s$ is public (sovereign) central bank digital currency (aka, CBDC).

Fiat reserves and digital currency can be stored across periods and dates, and fiat reserves are fungible with public digital currency for banks. $\bar{M}$, $\bar{M}^v$ and $\bar{M}^s$ are each an aggregate amount of outside money, with normalized per capita (unit mass of consumers) amounts at date τ defined as $M \equiv \frac{\bar{M}}{n^\tau}$ and $M_\tau^\iota \equiv \frac{\bar{M}^\iota}{n^\tau}$ for $\iota \in \{v, s\}$.

Goods market and nominal prices There is a goods market at each period $t \in \{0, 1, 2\}$ with fiat money as numeraire at all dates τ, and also for private digital currency as numeraire at dates $\tau \geq \tau'$. The goods market price is $P_{t,\tau}^\iota$, where $\iota \in \{v, s\}$ is the numeraire. Hence, prices $P_{t,\tau}^s$ refer to the amount of fiat money, which for dates $\tau \geq \tau'$ includes public digital currency, per unit of goods. For dates $\tau \geq \tau'$, prices $P_{t,\tau}^v$

refer to the amount of private digital currency per unit of goods. For dates $\tau < \tau'$, $P^v_{t,\tau}$ is not defined.

$X^\iota_{t,\tau} \equiv \frac{P^s_{t,\tau'}}{P^v_{t,\tau'}}$ is the exchange rate between digital currency $\iota \in \{v, s\}$ and fiat money at period t of date $\tau \geq \tau'$. Hence, $X^v_{t,\tau}$ is the quantity of fiat money per unit of private digital currency, and $X^s_{t,\tau} = 1$. For convenience of language, 'rate' is used interchangeably with 'return' in all contexts to refer to gross rate of return rather than net rate of return throughout the paper.

Banks and firms In order to simplify the presentation, the analysis throughout the paper including quantities is presented on a normalized per capita basis. Banks take deposits $D^\iota_{0,\tau}$ from consumers and lend $L^{F\iota}_{0,\tau}$ to firms, where $\iota \in \{v, s\}$ indicates that deposits and loans can be denominated in either fiat money or, for dates $\tau \geq \tau'$, private digital currency. Firms use loans to buy goods from consumers, and firms choose the allocation of goods to store and to invest in assets. At period $t = 1$, firms rollover an amount of borrowing $L^{F\iota}_{1,\tau}$. Banks can borrow $L^{CB}_{1,\tau}$ in fiat reserves or at $\tau \geq \tau'$ in public digital currency from the central bank. Throughout the paper, uppercase letters denote nominal variables, and lowercase letters denote real variables.

$R^{k\iota}_{t,\tau}$ is the return paid at $t \in \{1, 2\}$ on the type of deposit or loan $k \in K \equiv \{D, F, CB\}$, which correspond to deposits, loans to firms, and bank borrowing from the central bank, respectively. The value $\delta^k_{t,\tau} \leq 1$ is the fraction of the quantity actually repaid at $t \in \{1, 2\}$ of the deposit or loan type $k \in K$. For example, deposits pay a total return of $\delta^D_{t,\tau} D^\iota_{0,\tau} R^{D\iota}_{t,\tau}$ when withdrawn at $t \in \{1, 2\}$. If $\delta^k_{t,\tau} < 1$, the borrowing agent defaults at period t of date τ, which requires the borrowing agent to pay all revenues possible to maximize $\delta^k_{t,\tau}$.[32] If a bank takes a deposit in public or private digital currency, it must repay that digital currency in kind when withdrawn

[32] For simplicity, I assume that in case of a bank default $\delta^D_{t,\tau} < 1$ at period $t \in \{1, 2\}$, there is a pro rata sharing rule among deposits withdrawn at that period. Results do not change if there were instead any type of priority rule, such as with a sequential service constraint for deposit withdrawals at $t = 1$ in which some deposit withdrawals have no default, $\delta^D_{1,\tau} = 1$, and the remaining deposit withdrawals have a complete default, $\delta^D_{1,\tau} = 0$. A bank default on withdrawals at $t = 1$, $\delta^D_{1,\tau} < 1$, requires the bank to pay all of its revenues at $t = 1$ for withdrawals. This implies that the bank cannot rollover any lending to its firms, $L^{F\iota}_{1,\tau} = 0$. Hence, the bank will not have any revenues at $t = 2$, has a complete default at $t = 2$, $\delta^D_{2,\tau} = 0$, and cannot borrow from the central bank, $L^{CB}_{1,\tau} = 0$. Such a bank is referred to as liquidated at $t = 1$, since it has no loan assets after $t = 1$. Since the bank's firms cannot rollover any of their loans, these firms will also default at $t = 1$, $\delta^F_{1,\tau} < 1$. The firms must fully liquidate their assets, $a_{1,\tau} = a_{0,\tau}$, to sell goods and repay as much of their loans at $t = 1$ as possible.

or else default.

Since the aggregate state $(\lambda_\tau, r_{2,\tau})$ and a depositor's early or late type are not verifiable, there are incomplete markets in the form of standard short-term debt for the returns $R^{D\iota}_{1,\tau}$, $R^{D\iota}_{2,\tau}$ and $R^{F\iota}_{1,\tau}$on deposits and loans issued at $t = 0$. Returns $R^{F\iota}_{2,\tau}$ and $R^{CB}_{2,\tau}$ on $t = 1$ loans, and quantities and prices at $t \in \{1, 2\}$, are contingent on the aggregate state. For simplicity of notation, the state $(\lambda_\tau, r_{2,\tau})$ is suppressed in the writing of these dependent variables except where it is included for particular emphasis. In addition, the subscript for the generic date τ is omitted, except where it is included to refer to a particular non-generic date or to provide extra clarity when comparing a variable across different dates.

Outside money The model is developed to allow for a parsimonious representation of outside money, whether in the form of traditional fiat reserves, public digital currency, or private digital currency. The distinction of the regime with fiat reserves, before digital currency is introduced at date τ', is that there is not a form of outside money held by consumers or firms, such as in the form of paper currency outside of the banking system. This is motivated by the fact that it is much too costly for paper currency to be stored, secured, and transacted in markets on the large scale that is transacted in the economy through electronic bank payments. When digital currency is introduced starting date τ', the ability for consumers to use it for efficient payment transactions outside of the banking system is a key distinguishing feature of digital currency in the model.

2.3 Optimizations

The model is first presented with the assumption that there are no early deposit withdrawals at $t = 1$ by late consumers in order to examine the effects of fiat money and digital currency without the potential threat of bank runs. In section 5, this assumption is relaxed to study digital currency runs.

Consumers At period $t = 0$ of date τ, the representative consumer's budget constraint is

$$\textstyle\sum_\iota (D^\iota_0 + M^{C\iota}_0) X^\iota_0 \leq \sum_\iota (e^\iota_0 P^\iota_0 + \mathbf{1}_{\tau=\tau'} M^\iota) X^\iota_0, \tag{1}$$

where $\iota = s$ corresponds to units of fiat money for all dates τ, and $\iota = v$ corresponds to units of private digital currency that applies only for dates $\tau \geq \tau'$, in which case X_0^v converts units of private digital currency into fiat money. For $\iota = s$ and, for dates $\tau \geq \tau'$, $\iota = v$, the RHS of the inequality shows the consumer has proceeds of $e_0^\iota P_0^\iota$ from selling e_0^ι of her endowment goods for fiat money and for private digital currency, respectively. The consumer also receives M^ι digital currency at date τ', where $\mathbf{1}_{\{\cdot\}}$ is the indicator function. On the LHS of the inequality, the consumer deposits D_0^ι and (at dates $\tau \geq \tau'$) stores M_0^{Cs} public and M_0^{Cv} private digital currency.

Consumption for early and late consumers from goods bought with deposit withdrawals and digital currency can be expressed as:

$$\left.\begin{array}{ll} \text{early consumer at } t=1\text{:} & c_1 = \sum_\iota (\frac{\delta_1^D D_0^\iota R_1^{D\iota}}{P_1^\iota} + \frac{M_0^{C\iota}}{P_1^\iota}) \\ \text{late consumer at } t=1\text{:} & c_1^i = \sum_\iota (\frac{M_0^{C\iota} - M_1^{C\iota i}}{P_1^\iota}) \\ \text{late consumer at } t=2\text{:} & c_2^i = \sum_\iota (\frac{\delta_2^D D_0^\iota R_2^{D\iota}}{P_2^\iota} + \frac{M_1^{C\iota i}}{P_2^\iota}), \end{array}\right\} \tag{2}$$

where $M_1^{C\iota i}$ is the amount of digital currency stored from $t = 1$ to $t = 2$ by late consumer $i \in I$.

Consumers have expected utility

$$EU = E[\lambda u(c_1) + (1-\lambda) u(c_1^i + c_2^i)],$$

and have the following optimization:

$$\left.\begin{array}{l} \max\limits_{Q^{Ci}} EU \\ \text{s.t.:} \\ t\text{=0: Eq (1)} \\ \qquad \sum_\iota e_0^\iota \leq e_0 \\ \qquad M_1^{C\iota i} \leq M_0^{C\iota} \quad \forall (\lambda, r_2), \end{array}\right\} \tag{3}$$

with $(M_0^{C\iota}, M_1^{C\iota i}) \geq 0$ and $Q^{Ci} \equiv \{e_0^\iota, D_0^\iota, M_0^{C\iota}, M_1^{C\iota i}\}_{\lambda, r_2, \iota}$. The first inequality is the consumer's budget constraint at $t = 0$, and the last two inequalities are feasibility constraints.

Banks Because of free entry, the representative bank maximizes its depositors expected utility from the consumption that the bank's deposits provide at each date τ,

which is

$$EU^B = E\left[\lambda u\left(\sum_\iota \frac{\delta_1^D D_0^\iota R_1^{D\iota}}{P_1^\iota}\right) + (1-\lambda)u\left(\sum_\iota \frac{\delta_2^D D_0^\iota R_2^{D\iota}}{P_2^\iota}\right)\right].$$

The bank's optimization is:

$$\left.\begin{array}{l}
\max\limits_{Q^B} EU^B \\
\text{s.t.:} \\
t{=}0{:}\ \sum_\iota (L_0^{F\iota} + M_0^{B\iota}) X_0^\iota \le \sum_\iota (D_0^\iota + M_{2,\tau-1}^{B\iota}) X_0^\iota + \mathbf{1}_{\tau=0} M \\
t{=}1{:}\ \sum_\iota \lambda \delta_1^D D_0^\iota R_1^{D\iota} X_1^\iota \le \sum_\iota (\delta_1^F L_0^{F\iota} R_1^{F\iota} - L_1^{F\iota} + M_0^{B\iota} - M_1^{B\iota}) X_1^\iota + L_1^{CB} \quad \forall(\lambda, r_2) \\
t{=}2{:}\ \sum_\iota (1-\lambda) \delta_2^D D_0^\iota R_2^{D\iota} X_2^\iota \le \sum_\iota (\delta_2^F L_1^{F\iota} R_2^{F\iota} + M_1^{B\iota} - M_2^{B\iota}) X_2^\iota - \delta_2^{CB} L_1^{CB} R_2^{CB} \quad \forall(\lambda, r_2),
\end{array}\right\} \tag{4}$$

with $(M_1^{B\iota}, M_2^{B\iota}) \ge 0$ and

$$Q^B \equiv \{D_0^\iota, \{L_t^{F\iota}\}_{t\in\{0,1\}}, L_1^{CB}, \{\delta_t^D\}_{t\in\{1,2\}}, \delta_2^{CB}, \{M_t^{B\iota}\}_{t\in\{0,1,2\}}\}_{\lambda, r_2, \iota},$$

where at period t, the bank stores M_t^{Bs} fiat money reserves and (for dates $\tau \ge \tau'$) M_t^{Bv} private digital currency reserves. The three inequalities are the bank's budget constraints and must hold for all dates τ. At period $t = 0$, the bank's loans to firms and reserves stored are limited by deposits, $M_{2,\tau-1}^{B\iota}$ reserves stored at period $t = 2$ of the previous date $\tau - 1$, and initial fiat reserves M received at date $\tau = 0$. At periods $t = 1$ and $t = 2$, banks must meet deposit withdrawals out of net revenues from loans to firms and borrowing from the central bank.

Firms The representative firm consumes $c^F_{2,\tau}$, abbreviated as c^F_2, at $t = 2$ of date τ and maximizes profit in the form of expected consumption as follows:

$$\left.\begin{array}{l} \max_{Q^F} E[\sum_{\tau=0}^{\infty} c_2^F] \\ \text{s.t.:} \\ t{=}0\text{: } \sum_\iota [q_0^\iota P_0^\iota + M_0^{F\iota}] X_0^\iota \leq \sum_\iota (L_0^{F\iota} + M_{2,\tau-1}^{F\iota}) X_0^\iota \\ t{=}1\text{: } \sum_\iota \delta_1^F L_0^{F\iota} R_1^{F\iota} X_1^\iota \leq \sum_\iota (L_1^{F\iota} + q_1^\iota P_1^\iota + M_0^{F\iota} - M_1^{F\iota}) X_1^\iota \;\; \forall(\lambda, r_2) \\ t{=}2\text{: } \sum_\iota \delta_2^F L_1^{F\iota} R_2^{F\iota} X_2^\iota \leq \sum_\iota (q_2^\iota P_2^\iota + M_1^{F\iota} - M_2^{F\iota}) X_2^\iota \;\; \forall(\lambda, r_2) \\ \quad a_1 \leq a_0 \\ \quad g_0 + a_0 \leq \sum_\iota q_0^\iota \\ \quad \sum_\iota q_1^\iota \leq g_0 + a_1 r_1 - g_1 \;\; \forall(\lambda, r_2) \\ \quad \sum_\iota q_2^\iota \leq g_1 + (a_0 - a_1) r_2 - c_2^F \;\; \forall(\lambda, r_2), \end{array}\right\} \quad (5)$$

with $(g_1, a_1, M_1^{F\iota}, M_2^{F\iota}) \geq 0$, where q_t^ι is the quantity of goods bought (at period $t = 0$) and sold (at periods $t \in \{1, 2\}$) in the goods market for fiat money (including public digital currency at $\tau \geq \tau'$) for $\iota = s$ and for private digital currency at $\tau \geq \tau'$ for $\iota = v$, and where

$$Q^F \equiv \{\{g_t, a_t, L_t^{F\iota}\}_{t\in\{0,1\}}, \{\delta_t^F\}_{t\in\{1,2\}}, \{q_t^\iota, M_t^{F\iota}\}_{t\in\{0,1,2\}}\}_{\lambda, r_2, \iota, \tau}.$$

The first three inequalities are the firm's budget constraints, the latter four inequalities are feasibility constraints, and all constraints must hold for all dates τ.

Central bank The central bank's utility is that of the expected utility of consumers, except with the discount factor $\beta^{CB} \leq 1$ on period $t = 2$ consumption, which is expressed as

$$EU^{CB} = E[\lambda(c_1) + (1 - \lambda)u(c_1^i + \beta^{CB} c_2^i)].$$

If $\beta^{CB} < 1$, the central bank has a short-term bias with more concern about consumption, and hence output in the economy, at period $t = 1$ over that at period $t = 2$. The central bank can directly choose the nominal interest rate R_2^{CB} for lending fiat money to banks because of the central bank's monopoly power over fiat money. The

central bank optimization is

$$\left.\begin{array}{l} \max\limits_{R_2^{CB}} EU^{CB} \\ \text{s.t.:} \\ t{=}1\text{: } M_1^{CB} = L_1^{CB} \quad \forall(\lambda, r_2) \\ t{=}2\text{: } M_2^{CB} = -\delta_2^{CB} L_1^{CB} R_2^{CB} \quad \forall(\lambda, r_2), \end{array}\right\} \tag{6}$$

where M_1^{CB} is new fiat money the central bank creates to lend to banks at period $t = 1$, and M_2^{CB} is the negative of bank loan repayments, which reflects fiat money withdrawn from the economy.

Payments with fiat money and digital currency In order to focus on examining the potential efficiency and financial stability benefits of digital currency, as with fiat money, that derive from its role as an efficient means of payment and unit of account, electronic transaction payments using bank deposits or digital currency occur simultaneously within a period. I assume simultaneous digital transaction payments in order to shut down the channel for digital currency to have a positive value purely from a direct means-of-payment liquidity premium. A liquidity premium value for an outside money would be equal to the present value of future payment liquidity services for non-instantaneous transactions as in Bias et al. (2018b). More efficient payments imply a lower liquidity premium value for outside money. With the simplification of assuming instant bank and digital currency payments, there is no liquidity premium value.

Definition 1 *A market equilibrium is defined as the vector of prices and returns*

$$(\{\{P_t^\iota\}_{t\in\{0,1,2\}}, \{R_t^{D\iota}, R_t^{F\iota}\}_{t\in\{1,2\}}, R_2^{CB}\}_{\lambda,r_2,\iota,\tau}),$$

such that $\{R_2^{CB}\}_\tau$ *satisfies the central bank optimization (6), and at the optimizing quantities for consumers* $\{Q^{Ci}\}_\tau$*, banks* $\{Q^B\}_\tau$ *and firms* Q^F *given by optimizations (3), (4) and (5), respectively, and* $\{M_t^{CB}\}_{t\in\{1,2\},\tau}$ *given by the central bank optimization (6), markets clear at each date* τ*, and* $\forall(\lambda, r_2)$ *for* $t \in \{1, 2\}$*, for:*

(a) deposits: D_0^ι *for* $\iota \in \{v, s\}$*;*

(b) loans to firms: $L_t^{F\iota}$ *at* $t \in \{0, 1\}$ *for* $\iota \in \{v, s\}$*;*

(c) central bank loans to banks: L_1^{CB}*;*

(d) private digital currency: $\sum_{\kappa\in\{C,F,B\}} M^{\kappa v}_{t,\tau} = \mathbf{1}_{\tau\geq\tau'} M^v$ *at* $t \in \{0,1,2\}$*;*

(e) fiat money at $t \in \{0,1,2\}$*:*

$\sum_{\kappa\in\{C,F,B\}} M^{\kappa s}_{t,\tau} = M + \mathbf{1}_{\tau\geq\tau'} M^s + \sum_{\hat{\iota}=1}^{2} (\sum_{\hat{\tau}=0}^{\tau-1} M^{CB}_{\hat{\iota},\hat{\tau}} + \mathbf{1}_{\hat{\iota}\leq t} M^{CB}_{\hat{\iota},\tau})$*; and*

(f) goods at $t \in \{0,1,2\}$*:*

$t = 0$*:* $g_0 + a_0 = \sum_\iota e^\iota_0$,

$t = 1$*:* $\lambda c_1 + (1-\lambda)\int_{i\in I} c^i_1 = \sum_\iota q^\iota_1$,

$t = 2$*:* $(1-\lambda)\int_{i\in I} c^i_2 = \sum_\iota q^\iota_2$.

3 Fiat money

I initially analyze the baseline economy with only fiat money in the form of reserves to show the potential impact of distortionary fiat inflation, which provides the rationale for the introduction of digital currency in the next section. To provide a benchmark for the subsequent market equilibrium analysis, I first present the full-information, first best allocation.

3.1 First best

The planner's optimization is to maximize consumer's expected utility as follows:

$$\left.\begin{array}{l} \max\limits_{\{g_t,a_t\}_{t\in\{0,1\}}} \quad EU = E\left[\lambda u(c_1) + (1-\lambda)\, u(c^i_1 + c^i_2)\right] \\ \text{s.t.:} \\ t\text{=0: } g_0 + a_0 \leq e_0 \\ t\text{=1: } \lambda c_1 \leq g_0 - g_1 + a_1 r_1 \quad \forall(\lambda, r_2) \\ t\text{=2: } (1-\lambda)\,(c^i_1 + c^i_2) \leq (a_0 - a_1) r_2 + g_1 \quad \forall(\lambda, r_2) \end{array}\right\} \tag{7}$$

The first-order conditions and binding constraints for the planner's optimization give optimal consumption according to

$$E[u'(c^*_1)] = E[r_2 u'(c^*_2)] \tag{8}$$

$$c_1 = c^*_1 = \tfrac{g^*_0 - g^*_1 + a^*_1 r_1}{\lambda} \tag{9}$$

$$c^i_2 = c^*_2 = \tfrac{(a^*_0 - a^*_1) r_2 + g^*_1}{1-\lambda} \tag{10}$$

$$c^i_1 = 0. \tag{11}$$

The first line above gives the Euler equation showing that in expectation, the ratio of marginal utilities between $t = 1$ and $t = 2$ is equal to the marginal rate of transformation r_2.

Optimal liquidity risk-sharing between early and late consumers decreases consumption risk, with expected consumption $E[c_1^*] > 1$ and $E[c_2^*] \in (E[c_1^*], \bar{r}_2)$. This is implemented with an optimal quantity of $t = 0$ storage, g_0^*, that is greater than the endowment of the expected fraction of early consumers, $\bar{\lambda}$.

The optimal consumption for early and late consumers, c_1^* and c_2^*, depend on the optimal amount of goods (g_1^*) stored from $t = 1$ to $t = 2$ and the optimal asset liquidation (a_1^*) at $t = 1$, which both depend upon the realization of the aggregate state (λ, r_2). The following proposition shows that $g_1^* > 0$ when λ and r_2 are relatively low, written as $\lambda < \check{\lambda}(r_2)$, and $a_1^* > 0$ when λ and r_2 are relatively high, written as $\lambda > \hat{\lambda}(r_2)$.

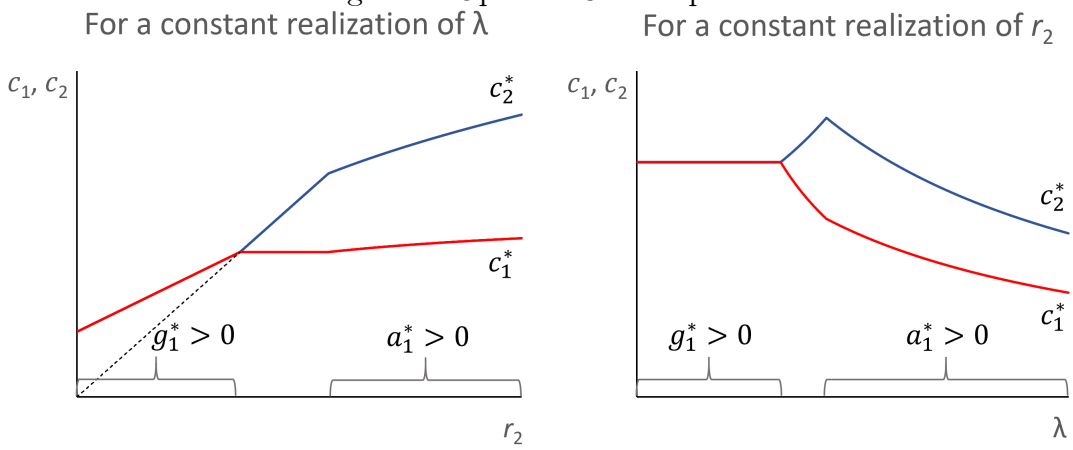
Proposition 1 *The optimal amount of storage at $t = 1$ is positive, $g_1^* > 0$, when there are relatively few early consumers and low asset returns. Conversely, the optimal amount of asset liquidation at $t = 1$ is positive, $a_1^* > 0$, when there are relatively many early consumers and high asset returns.*

A comparison of optimal consumption for early and late consumers is illustrated in the two diagrams in Figure 1. As r_2 increases for a constant realization of λ, and as λ increases for a constant realization of r_2, there is initially a decreasing amount of positive storage at $t = 1$ to provide equal consumption $c_2^* = c_1^*$, then no $t = 1$ storage or liquidation, and finally an increasing amount of $t = 1$ liquidation to provide a partial transfer of late consumers' increasing consumption to early consumers.

3.2 Fiat reserves equilibrium

At each date $\tau < \tau'$, banks hold fiat reserves and lend L_0^{Fs} to firms, who purchase endowment goods from the new generation of consumers at the period $t = 0$ price level P_0^s. The initial price level at period $t = 0$ of date $\tau = 0$ is not determined and without loss of generality is normalized to one: $P_{0,0}^s \equiv 1$. Consumers deposit their revenues as D_0. At periods $t = 1$ and $t = 2$, consumers withdraw deposits and buy

Figure 1: Optimal Consumption



goods from firms at equilibrium prices

$$P_1^s(\lambda, r_2) = \frac{\lambda \delta_1^D D_0^s R_1^{Ds}}{q_1^s} \tag{12}$$

$$P_2^s(\lambda, r_2) = \frac{(1-\lambda) \delta_2^D D_0^s R_2^{Ds}}{q_2^s}. \tag{13}$$

The price levels reflect the amount of money supplied by consumers for purchasing goods divided by the amount of goods sold by firms within each period.

Inflation Within date τ, inflation between periods is defined as follows:

$$\Pi_{1,\tau}^s(\lambda, r_2) \equiv \frac{P_{1,\tau}^s}{P_{0,\tau}^s} \quad \text{inflation between periods } t = 0 \text{ and } t = 1$$

$$\Pi_{2,\tau}^s(\lambda, r_2) \equiv \frac{P_{2,\tau}^s}{P_{1,\tau}^s} \quad \text{inflation between periods } t = 1 \text{ and } t = 2.$$

Fiat money with first best allocation First consider the case of $\beta^{CB} = 1$, in which the central bank does not have a short-term bias.

Proposition 2 *If the central bank does not have short-term bias, $\beta^{CB} = 1$, the market equilibrium provides the optimal first best consumption c_1^* and c_2^* with no bank defaults, $\delta_t^D = 1$, for all realizations of (λ, r_2).*

Since deposits pay out nominal amounts, the bank can pay fixed promises in terms of money as numeraire with no bank defaults, $\delta_t^D = 1$, yet depositors' consumption can flexibly respond to aggregate real and liquidity shocks in the economy through elastic prices, which reflects an elastic real value of fiat money. The real return per

unit on deposits provides consumption contingent on the aggregate state (λ, r_2) for early and late types, $c_1 = \frac{D_0^s R_1^{Ds}}{P_1^s} = \frac{q_1^s}{\lambda}$ and $c_2^i = \frac{D_0^s R_2^{Ds}}{P_2^s} = \frac{q_2^s}{1-\lambda}$, respectively.

Optimal consumption The first key result for the market to provide optimal consumption is that at $t = 0$, firms store the optimal amount of real liquidity, $g_0 = g_0^*$. The optimal storage follows from the Euler equation from the bank optimization of its depositors' expected utility for the provision of optimal liquidity for early consumers, $E[u'(c_1^*)] = E[r_2 u'(c_2^*)]$, which is equivalent to Euler equation for the planner's optimization. At $t = 0$, the expected real return on bank loans to firms at $t = 1$ is equal to the expected return on assets, $E[\frac{R_2^{Fs}}{\Pi_2^s}] = \bar{r}_2$, which is greater than the implicit expected real return paid on deposit withdrawals at $t = 2$ relative to $t = 1$: $\frac{E[c_2^*]}{E[c_1^*]} < \frac{E[u'(c_1^*)]}{E[(c_2^*)]} = \bar{r}_2$.

The second key result is that the market provides the optimal rationing of goods between early and late consumers through the optimal quantity of goods sold by firms, $q_1^{s*} = \frac{g_0^* + a_1^* r_1 - g_1^*}{\lambda} = \frac{c_1^*}{\lambda}$ and $q_2^{s*} = \frac{(a_0^* - a_1^*) r_2 + g_1^*}{1-\lambda} = \frac{c_2^*}{1-\lambda}$, due to the price mechanism. The real rate on loans to firms between periods $t = 1$ and $t = 2$ is $r_2^{Fs} \equiv \frac{R_2^{Fs}}{\Pi_2^s}$. First order conditions for the firm's optimization require that if there is positive storage at $t = 1$, $g_1 > 0$, then the real rate must equal one and there is no liquidation, $r_2^{Fs} = 1$ and $a_1 = 0$. Conversely, if there is positive liquidation of the long asset, $a_1 \in (0, a_0)$, then the real rate r_2^{Fs} must equal $\frac{r_2}{r_1}$, which is the marginal rate of transformation for the illiquid asset not being liquidated (r_2) relative to being liquidated (r_1), and hence there is no storage at $t = 1$, $g_1 = 0$.

As a result, if there is a moderate real rate $r_2^{Fs} \in (1, \frac{r_2}{r_1})$ in equilibrium, there is no storage or liquidation at $t = 1$, $g_1 = a_1 = 0$. Whereas, if there is a high real rate $r_2^{Fs} = \frac{r_2}{r_1}$ in equilibrium, there is positive liquidation and no $t = 1$ storage, $a_1 > 0$ and $g_1 = 0$.

In particular, for a moderate realization of (λ, r_2) given by $\lambda \in [\check{\lambda}(r_2), \hat{\lambda}(r_2)]$, the equilibrium price levels at $t = 1$ and $t = 2$ are moderate, with $P_1^s = \frac{\lambda D_0^s R_1^{Ds}}{g_0^*}$ and $P_2^s = \frac{(1-\lambda) D_0^s R_2^{Ds}}{a_0^* r_2}$, and the real rate is the optimal $r_2^{Fs^*} \in [1, \frac{r_2}{r_1}]$. Firms sell at $t = 1$ all of their goods stored from $t = 0$ and sell at $t = 2$ the returns on all their assets. For a low realization of (λ, r_2) given by $\lambda < \check{\lambda}(r_2)$, there is downward pressure on P_1^s and upward pressure on P_2^s, with optimal real rate $r_2^{Fs^*} = 1$. With fewer early consumers, the amount of inside money spent for goods is reduced at $t = 1$ and increased at

$t = 2$. With lower returns, fewer goods produced by assets are available to sell at $t = 2$. Firms respond to these market prices by storing the optimal amount g_1^* of their goods at $t = 1$ to sell at $t = 2$, which provides for equal consumption among early consumers withdrawing at $t = 1$ and late depositors withdrawing at $t = 2$. Conversely, for a high realization of (λ, r_2) given by $\lambda > \hat{\lambda}(r_2)$, there is relative upward pressure on P_1^s and downward pressure on P_2^s, with optimal real rate $r_2^{Fs^*} = \frac{r_2}{r_1}$, where $\frac{u'(c_1^*)}{u'(c_2^*)} = \frac{r_2}{r_1}$. Firms respond by liquidating the optimal amount a_1^* of their assets to sell additional goods at $t = 1$. For all realizations of (λ, r_2), firms have zero consumption: $c_2^F = 0$.

Central bank rate First order conditions for the bank's optimization require that the rate between periods 1 and 2 on loans to firms, R_2^{Fs}, and the central bank rate, R_2^{CB}, are equal. According to the central bank's optimization with $\beta = 1$, the central bank is passive and optimally sets its interest rate equal to the optimal equilibrium loan rate, $R_2^{CB^*} = R_2^{Fs^*}$, and there is no borrowing by banks from the central bank, $L_1^{CB} = 0$.

Corollary 1 *If the central bank does not have a short-term bias, $\beta^{CB} = 1$, expected and average inflation across periods and dates is zero: $E[\Pi_{t,\tau}^{s^*}] = 1$ and $E[\frac{P_{t,\tau+1}^{s^*}}{P_{t,\tau}^{s^*}}] = 1$ for $t \in \{0, 1, 2\}$.*

Central bank fiat inflation Central bank discretion over the supply of fiat money presents the potential of distortionary inflation if the central bank has a bias for higher short-term consumption and output that arises from a lower discount factor $\beta^{CB} < 1$ than consumers' discount factor at period $t = 2$, which has been implicitly set equal to one. The central bank short-term bias can take two different forms, which are analyzed in turn. One form is that the central bank's bias comes as a surprise to the public at period $t = 1$, after asset investment decisions are made at period $t = 0$. The second form is that the central bank's bias is known by the public at period $t = 0$.

For the first form of bias, the public expects at $t = 0$ that the central bank has a discount factor $\beta^{CB} = 1$ and will set its policy rate at R^{CB^*}, and firms choose a_0^* as their asset investment. At $t = 1$, the central bank unexpectedly sets a higher nominal rate $\hat{R}^{CB} > R_2^{CB^*}$ to maximize consumers' expected utility with the lower discount factor $B^{CB} < 1$, with a real central bank rate of $r_2^{CB} \equiv \frac{R_2^{CB}}{\Pi_2^s}$. For $g_1^* = 0$,

increasing $\hat{R}_2^{CB}$ above $R_2^{CB^*}$ implies that firms excessively liquidate assets at $t = 1$, where $\hat{a}_1(\hat{R}_2^{CB}) > a_1^*(R_2^{CB^*})$ and $\hat{r}_2^{Fs} = \hat{r}_2^{CB} = \frac{r_2}{r_1}$.

For the second form of central bank bias, the public knows the central bank's discount factor $\beta^{CB} < 1$. Rather than a greater amount of output at $t = 1$ through excessive asset liquidation at $t = 1$, firms in anticipation instead store excessive goods and hold lower investment at $t = 0$ than the first best: $\hat{a}_0 < a_0^*$.

Proposition 3 *For either an unexpected or expected central bank short-term bias of $\beta^{CB} < 1$, there is distortionary inflation at $t = 2$ of $\hat{\Pi}_2^s > \Pi_2^{s^*}$ through the central bank setting $\hat{R}_2^{CB} > R_2^{CB^*}$ at $t = 1$, which increases output at $t = 1$ to $\hat{q}_1^s > q_1^{s*}$ and decreases output at $t = 2$ to $\hat{q}_2^s < q_2^{s*}$. Early consumers receive higher consumption than optimal, $\hat{c}_1 \geq c_1^*$ with $E[\hat{c}_1] \geq E[c_1^*]$, and late consumers receive lower consumption than optimal, $\hat{c}_2^i \leq c_2^*$ with $E[\hat{c}_2^i] < E[c_2^*]$.*

I proceed by assuming that the central bank's discount factor $\beta^{CB} \leq 1$ is known by the public at $t = 0$, such that distortionary fiat inflation is fully anticipated when the central bank has a short-term bias with $\beta^{CB} < 1$.

4 Digital currency

In this section, I first analyze the equilibrium with only a public digital currency to establish its equivalence to fiat reserves. Then, private digital is included along with public digital currency.

4.1 Public digital currency

At period $t = 0$ of date τ', consumers receive M^s public digital currency in addition to $e_0 P_{0,\tau}^s$ revenues from selling their endowment goods. From these proceeds, consumers allocate the amount of money to store as digital currency, $M_{0,\tau'}^{Cs}$, and to hold as deposits, $D_{0,\tau'}^s$.

The real return in terms of consumption from holding deposits is expressed, equivalent to the case of fiat money, as $\frac{R_1^{Ds}}{P_1^s}$ for early consumers and $\frac{R_2^{Ds}}{P_2^s}$ for late consumers. Whereas, the real return in terms of consumption from holding public digital currency is $\frac{1}{P_1^s}$ for early consumers and $\max\{\frac{1}{P_1^s}, \frac{1}{P_2^s}\}$ for late consumers.

If the central bank has a short-term bias, the impact of distortionary fiat inflation between periods $t = 1$ and $t = 2$ occurs through lower than optimal prices at $t = 1$, $\hat{P}_1^s < P_1^{s^*}$, and higher than optimal prices at $t = 2$, $\hat{P}_2^s > P_2^{s^*}$, which results from the distortionary inflation $\hat{\Pi}_2^s > \Pi_2^{s^*}$. This distortionary inflation impacts the periods $t = 1$ and $t = 2$ real value of public digital currency stored by consumers at $t = 0$ and deposits made at $t = 0$ in an equivalent manner. Hence, regardless of whether or not the central bank has a short-term bias, consumers prefer to deposit their entire proceeds at period $t = 0$ and not hold public digital currency since deposits pay the nominal return R_t^{Ds}, which is greater than the return of one on storing public digital currency.

Lemma 1 *With public digital currency, consumers continue to hold bank deposits rather than hold public digital currency directly.*

Effect of public digital currency on inflation Since consumers deposit their public digital currency, along with their proceeds from selling their endowment goods, the size of consumer deposits at date τ' increases relative to prior dates by the amount of the public digital currency: $D_{0,\tau'}^s = e_0 P_{0,\tau'}^s + M^s$. The increase in the nominal size of deposits leads to an increase in the nominal amount of money withdrawn and paid for goods at periods $t = 1$ and $t = 2$, which results in an increase in expected prices $E[P_{1,\tau}^s]$ and $E[P_{2,\tau}^s]$ at date $\tau = \tau'$ relative to at dates prior to τ' by the amount of the public digital currency M^s. This higher general price level continues at future dates $\tau > \tau'$. Hence, the effect of public digital currency introduced at date τ' is a one-time increase in inflation because of the increase in overall outside fiat money. However, this inflation is not distortionary because the relative price level between periods $t = 1$ and $t = 2$ of date τ' and future dates, which is the inflation rate $\Pi_{t,\tau}^s$ for periods $t \in \{1, 2\}$ at all dates $\tau \geq \tau'$, is not effected.

Lemma 2 *Public digital currency creates a one-time inflation through a higher general price level at date τ' that is not distortionary.*

Public digital currency held by consumers as bank deposits is equivalent to fiat money deposits, which provide liquidity risk sharing but bear the cost of distortionary inflation when the central bank has a short-term bias, $\beta^{CB} < 1$.

Since public digital currency is deposited at banks, it ends up being held as an increase of overall fiat money reserves in the banking system. At period $t = 0$ of date τ', after loans $L^{Fs}_{0,\tau'}$ to firms, banks store $M^{Bs}_{0,\tau'}$ in the form of fiat and public digital currency reserves out of the deposits received and fiat reserves held from prior dates. Since in equilibrium, consumer revenues from selling goods are equal to the amount of loans firms spend to buy goods, $e_0 P^s_{0,\tau'} = L^{Fs}_{0,\tau'}$, banks retain at period $t = 0$ overall fiat money reserves equal to initial fiat reserves and the new public digital currency. At each period $t = 1$ and $t = 2$, bank revenues from loan repayments (net of rollover lending to firms at $t = 1$) equal the amount of deposit withdrawals. Hence, banks hold their period $t = 0$ fiat money, $M^{Bs}_{0,\tau}$, throughout date τ' and into the following and future dates $\tau > \tau'$.

4.2 Private digital currency

Now consider private as well as public digital currency introduced at date τ'.

4.2.1 Private digital currency without deposits

First, I analyze the economy with the assumption that private digital currency is held outside of the banking system. At period $t = 0$ of date τ', consumers receive M^v private digital currency, M^s public digital currency, and revenues from selling their endowment goods. From these proceeds, consumers allocate the amount of money to store as private digital currency, $M^{Cv}_{0,\tau'}$, and to hold as fiat money deposits, $D^s_{0,\tau'}$, since fiat money deposits dominate holding public digital currency directly. Firms use their loans from banks to buy consumer goods in fiat money. At periods $t = 1$ and $t = 2$, firms sell to consumers a portion of their output goods for fiat money to repay bank loans and remaining output goods for private digital currency, which firms hold into future dates. At these dates $\tau > \tau'$, firms use a combination of their holdings of private digital currency along with fiat money loans from banks to buy endowment goods from consumers at period $t = 0$. At periods $t = 1$ and $t = 2$, firms again sell output goods for fiat money to repay loans and for private digital currency to retain.

The real return in terms of consumption from holding private digital currency is $\frac{1}{P^v_1}$ for early consumers and $\max\{\frac{1}{P^v_1}, \frac{1}{P^v_2}\}$ for late consumers. The private digital

currency precludes central bank discretion and distortionary inflation for private digital currency prices. However, private digital currency held by consumers does not provide the liquidity risk sharing of bank deposits.

Proposition 4 *For private digital currency held by consumers outside of the banking system, expected consumption is* $E[c_1] = 1$ *and* $E[c_2^i] = \bar{r}_2$*, with* $c_1 < c_1^*$ *and* $c_2^i > c_2^*$*. Holding private digital currency avoids fiat inflation but does not benefit from bank liquidity risk sharing.*

For firms to be willing to sell goods for private digital currency at dates $\tau \geq \tau'$, a deflationary price level is required across dates, which provides a sufficient return for firms to hold private digital currency as a store of value.

Corollary 2 *The private digital currency price level must have an expected decrease equal to* $\bar{r}_2$ *across dates to provide a sufficient return for holding it:* $E[\frac{P^v_{t,\tau+1}}{P^v_{t,\tau}}] = \frac{1}{\bar{r}_2}$ *for periods* $t \in \{0, 1, 2\}$ *and dates* $\tau \geq \tau'$*.*

As a result, the exchange rate for the fiat value of private digital currency, $X^v_{t,\tau} = \frac{P^s_{t,\tau'}}{P^v_{t,\tau'}}$, increases by $\bar{r}_2$ in expectation across dates: $E[\frac{X^v_{t,\tau+1}}{X^v_{t,\tau}}] = \bar{r}_2$ for periods $t \in \{0, 1, 2\}$ and dates $\tau \geq \tau'$.

4.2.2 Private digital currency with bank deposits

Next, I examine the economy without the assumption that private digital currency is held outside of the banking system. In addition to fiat and public digital currency deposits, banks can take deposits and make loans that are denominated in private digital currency.

At period $t = 0$ of date τ', consumers have the additional option of making deposits $D^v_{0,\tau'}$ in private digital currency, and banks have the additional option of lending $L^v_{0,\tau'}$ in private digital currency to firms. Consumers with private digital currency deposits receive optimal consumption with liquidity risk sharing. Thus, consumers prefer to hold private digital currency in the form of bank deposits rather than storing it directly.

Proposition 5 *With private digital currency, consumers hold bank deposits denominated in private digital currency rather than holding private digital currency directly.*

Banks operate by holding private digital currency as a form of private reserves in a similar manner as holding public digital currency as fiat reserves.

Corollary 3 *Banks are not displaced by private or public digital currency.*

4.2.3 Private digital currency versus fiat money

If the central bank does not have a short-term bias, with $\beta^{CB} = 1$, consumers are indifferent between holding bank deposits denominated in private digital currency or fiat money. However, if the central bank does have a short-term bias, with $\beta^{CB} < 1$ that leads to distortionary fiat inflation, consumers prefer to hold only private digital currency deposits, which provide optimal consumption: $c_1 = c_1^*$, $c_2^i = c_2^*$. Fiat money and public digital currency is not held and is driven out by private digital currency.

Proposition 6 *Private digital currency drives out fiat money and public digital currency if the central bank has a short-term bias that creates distortionary fiat inflation.*

5 Digital runs

With private or public digital currency bank deposits, there is also a threat of digital currency runs that is now considered. The threat is that late consumers may withdraw early and store digital currency outside of the banking system at $t = 1$. Excessive withdrawals of digital currency can deplete the banking system and cause bank defaults at $t = 1$, which require banks to liquidate by not rolling over loans to firms. The analysis proceeds by analyzing the threat of runs, followed by the analysis of the central bank acting as lender of last resort.

5.1 Early withdrawals by late consumers

The model is updated to include the potential for early withdrawals by late consumers.

Late consumers Late consumer $i \in I$ makes an early withdrawal fraction $w^{\iota i} \in [0, 1]$ of her deposit D_0^ι at $t = 1$ and withdraws the remaining fraction $(1 - w^{\iota i})$ of her deposit D_0^ι at $t = 2$. For dates $\tau \geq \tau'$, of the late consumer's early withdrawal

at period $t = 1$, $M_1^{C\iota wi}$ is the withdrawal and storage quantity of digital currency $\iota \in \{v, s\}$, and the remainder of the early withdrawal is used to buy goods at $t = 1$. Consumption for late consumer $i \in I$ from goods bought at $t = 1$ and at $t = 2$ is updated as

$$\text{late consumer} \qquad t = 1: \;\; c_1^i = \sum\nolimits_\iota \left(\frac{\boxed{w^{\iota i}\delta_1^D D_0^\iota R_1^{D\iota} - M_1^{C\iota wi}}}{P_1^\iota} + \frac{M_0^{C\iota} - M_1^{C\iota i}}{P_1^\iota} \right)$$

$$\text{late consumer} \qquad t = 2: \;\; c_2^i = \sum\nolimits_\iota \left[\frac{(1 - \boxed{w^{\iota i}})\delta_2^D D_0^\iota R_2^{D\iota} + \boxed{M_1^{C\iota wi}}}{P_2^\iota} + \frac{M_1^{C\iota i}}{P_2^\iota} \right],$$

where the boxes indicate added terms. For late consumer $i \in I$, the amount of digital currency stored from $t = 1$ to $t = 2$ includes $M_1^{C\iota i} \leq M_0^{C\iota}$ of initial digital currency stored at $t = 0$ and $M_1^{C\iota wi} \in [0, w^{\iota i}\delta_1^D D_0^\iota R_1^{D\iota}]$ of digital currency withdrawn at $t = 1$.

Withdrawal strategy The *withdrawal strategy* for late consumer $i \in I$ is

$$\sigma^i \equiv \{w^{\iota i}(\lambda, r_2), M_1^{C\iota wi}(\lambda, r_2)\}_\iota.$$

The joint set of withdrawal strategies for all late consumers $i \in I$ is defined as the *withdrawal set* $\sigma \equiv \{\sigma^i\}_{i \in I}$.

Banks The budget constraints for banks at $t = 1, 2$ are updated with new expressions in boxes to reflect early withdrawals by late consumers as:

$$\begin{aligned}
t{=}1: \qquad & \textstyle\sum_\iota [\lambda + \boxed{(1 - \lambda)\int_{i \in I} w^{\iota i}}]\delta_1^D D_0^\iota R_1^{D\iota} X_1^\iota \\
\leq \;\; & \textstyle\sum_\iota (\delta_1^F L_0^{F\iota} R_1^{F\iota} - L_1^{F\iota} + M_0^{B\iota} - M_1^{B\iota}) X_1^\iota + L_1^{CB} \quad \forall (\lambda, r_2) \\
t{=}2: \qquad & \textstyle\sum_\iota (1 - \lambda)\boxed{(1 - \int_{i \in I} w^{\iota i})}\delta_2^D D_0^\iota R_2^{D\iota} X_2^\iota \\
\leq \;\; & \textstyle\sum_\iota (\delta_2^F L_1^{F\iota} R_2^{F\iota} + M_1^{B\iota} - M_2^{B\iota}) X_2^\iota - \delta_2^{CB} L_1^{CB} R_2^{CB} \quad \forall (\lambda, r_2).
\end{aligned}$$

Definition 2 *A Nash equilibrium at date τ of the late consumers' strategic withdrawal game is defined as the withdrawal set*

$$\{\sigma^i | \{\sigma^{i'}\}_{i' \in I}\}_{i \in I},$$

which is the set of withdrawal strategies σ^i for each late consumer $i \in I$, and where σ^i for each $i \in I$ is a best response given $\{\sigma^{i'}\}_{i' \in I}$, the withdrawal strategies of all other late consumers $i' \in I$.

In order to distinguish the threat that digital currency poses in the form of runs, it is important to first provide the contrasting result of financial stability that arises in the economy due to nominal prices and bank deposits.

Financial stability Consider a withdrawal set σ in which there are no early withdrawals by late consumers: $w^{\iota i} = M_1^{C\iota w i} = 0$ for $\iota \in \{v, s\}$ and all $i \in I$. The elastic value of fiat reserves, as well as public and private digital currency, enables elastic nominal prices in the economy, which supports a financial system that can create optimal asset and liquidity risk sharing and enhances financial stability of the banking system.

The elasticity of the price level at $t = 1$ and $t = 2$ reflects the elastic value of a digital currency, as with fiat reserves, since the real value of the digital currency at each period is the inverse of the price level. This elastic value of the digital currency that can provide the optimal allocation of consumption also enhances financial stability against two primary risks inherent in the banking system. One risk is solvency-based bank runs from the potential insolvency of the banking system in the case of low real returns on assets, r_2. The second risk is liquidity-based bank runs from the potential illiquidity of the banking system in the case of a large fraction of early consumers, λ.

First, consider the risk of insolvency in the case of low realizations of r_2. P_2^ν increases due to the reduction in goods available to sell at $t = 2$. This leads firms to hold over goods from $t = 1$ to sell at $t = 2$, such that late consumers do not receive any greater consumption by running the bank to buy goods at $t = 1$. Moreover, banks are effectively hedged on their nominal deposit liabilities at $t = 2$. The equilibrium price level at $t = 2$ remains elevated even with the counterbalancing effect of firms selling more goods at $t = 2$. The elevated price level implies that the real cost of banks' $t = 2$ deposit liabilities falls enough that banks do not default.

Second, consider the risk of the bank defaulting when there is a large realization of early consumers, λ. P_1^ν increases from the larger amounts of money spent for goods at $t = 1$. This leads firms to liquidate a greater amount of assets to sell additional goods at $t = 1$. While additional goods sold provides a partial counterbalancing effect on the price level, P_1^ν is still sufficiently elevated such that firms do not default on their loans to banks, and banks do not default on paying withdrawals. Banks continue to rollover loans to firms, which enables firms to only liquidate assets to the extent that

it is profit-maximizing according to selling goods at $t = 1$ relative to at $t = 2$. A marginal late consumer would not prefer to withdraw to buy goods at $t = 1$ because of the higher nominal deposit return as well as relatively lower price level P_2^ι at $t = 2$.

Proposition 7 *For all realizations of* (λ, r_2) *at each date* τ*, there exists a Nash equilibrium without bank runs:*

$$w^{\iota i} = M_1^{C\iota w i} = 0 \qquad \textit{for all } \iota \in \{v, s\} \textit{ and all } i \in I.$$

5.2 Digital currency runs

I now turn to the threats at date $\tau \geq \tau'$ banks face of withdrawal runs, with $w^{\iota i} = 1$ for all $i \in I$, that may take two different forms. The first threat comes from late consumers running the bank to secure real consumption at $t = 1$. In this case, late consumers do not withdraw any digital currency to store at $t = 1$ for buying goods at $t = 2$: $M_1^{C\iota w i} = 0$. The second threat comes from late consumers taking a part or all of their early withdrawals in the form of digital currency to store at $t = 1$ and buy goods at $t = 2$: $M_1^{C\iota w i} \in (0, \delta_1^D D_0^\iota R_1^{D\iota}]$. These withdrawal run threats are first considered in the absence of the central bank as lender of last resort, in which case $L_1^{CB} \equiv 0$.

For $w^{\iota i} = 1$, the price level at periods 1 and 2 depending on $M_1^{C\iota w i}$ are given in the following table:

$$\begin{array}{lll} & \underline{M_1^{C\iota w i} \in [0, \delta_1^D D_0^\iota R_1^{D\iota}]} & \underline{M_1^{C\iota w i} = \delta_1^D D_0^\iota R_1^{D\iota}} \\ P_1^\iota = & \frac{\delta_1^D D_0^\iota R_1^{D\iota} - \int_{i \in I} (1-\lambda) M_1^{C\iota w i}}{q_1^\iota} & \frac{\lambda \delta_1^D D_0^\iota R_1^{D\iota}}{q_1^\iota} \\ P_2^\iota = & \frac{\int_{i \in I} (1-\lambda) M_1^{C\iota w i}}{q_2^\iota} & \frac{(1-\lambda) \delta_1^D D_0^\iota R_1^{D\iota}}{q_2^\iota}. \end{array}$$

Under the first threat, with $w^{\iota i} = 1$ and $M_1^{C\iota w i} = 0$, all late consumers run on the banking system in order to buy goods at $t = 1$. Similar to above, the impact would be an increase in P_1^ι, which would lead firms to liquidate a greater amount of assets than otherwise in order to sell additional goods at $t = 1$. While additional goods sold would provide a partial counterbalancing effect on the price level, P_1^ι would still be sufficiently elevated such that firms would not default on their loans to banks, and banks would not default on paying withdrawals. Banks could continue to rollover loans to firms, which enables firms to only liquidate assets to the extent that it is

profit-maximizing for selling goods at $t = 1$ relative to at $t = 2$. A marginal late consumer would prefer to deviate from the strategy of withdrawing to buy goods at $t = 1$ in order to withdraw instead at $t = 2$ for the higher nominal deposit return as well as relatively lower price level P_2^ι. Thus, a marginal late consumer who deviates and withdraws instead at $t = 2$ has greater consumption. Hence, with $M_1^{C\iota wi} = 0$, all late consumers would prefer to withdraw at $t = 2$, and such liquidity-based runs do not occur in equilibrium The outcome of no bank runs, $w^{\iota i} = M_1^{C\iota wi} = 0$ for all $i \in I$, is a Nash equilibrium, and there are no defaults: $\delta_t^k = 1$ for all $k \in K,\ t \in \{1, 2\}$.

Under the second threat, with $w^{\iota i} = 1$ and $M_1^{C\iota wi} \in (0, \delta_1^D D_0^\iota R_1^{D\iota}]$, the withdrawal run equilibrium may exist. At date $\tau \geq \tau'$, the bank defaults at $t = 1$ if $(1-\lambda)M_1^{C\iota wi} > M_0^{B\iota}$. In particular, for $M_1^{C\iota wi} = \delta_1^D D_0^\iota R_1^{D\iota}$, this bank default condition is

$$(1 - \lambda)D_0^\iota R_1^{D\iota} > M_0^{B\iota},$$

which can be simplified as

$$\lambda < \tfrac{1}{1+m_{0,\tau}^\iota} \in (\tfrac{1}{2}, 1)$$

where $m_{0,\tau}^\iota \equiv \frac{M^\iota}{P_{0,\tau}^\iota} \in (0, 1)$ is defined as the real value at date $\tau \geq \tau'$ of the digital currency M^ι introduced at date τ'. Counterintuitively, a withdrawal run equilibrium can only occur at dates when there is a sufficiently low realization λ of early consumers. This is because with a greater amount of late consumers, there is a larger amount of digital currency withdrawals under a withdrawal run threat at $t = 1$ that has greater ability to deplete the bank, cause a bank default, and enable the withdrawal run threat to sustain as an equilibrium run.

Proposition 8 *With private or public digital currency deposits at date $\tau \geq \tau'$, for a small realization of early consumers $\lambda < \frac{1}{1+m_{0,\tau}^\iota}$, there exists a digital currency run Nash equilibrium in the form of digital currency withdrawals by late consumers that create a complete liquidation of the banking system in absence of a lender of last resort.*

5.3 Central bank as lender of last resort

The central bank has the ability and discretion to create an additional quantity of the supply of fiat money, which gives the central bank a natural monopoly over the

outside supply of liquidity available to banks. Because of this, the central bank has the unique ability to act as lender of last resort to banks with public digital currency deposits by issuing an additional quantity of public digital currency that is lent to banks facing runs at $t = 1$.

Regardless of the seniority of the central bank's loans to banks, the central bank can create and lend large enough amounts to such illiquid banks to ensure they do not default at $t = 1$ and $t = 2$. Hence, the the central bank does not face any risk of banks defaulting on the loans. Borrowing banks can repay the loans, comprised of outside digital currency at $t = 1$, in kind at $t = 2$ with public digital currency received from their returns on loans to firms.

The withdrawal run threat on a bank is that all late consumers $i \in I$ withdraw and store digital currency at $t = 1$: $w^{\iota i} \in (0, 1]$ and $M_1^{C\iota wi} = (0, w^{\iota i}\delta_1^D D_0^\iota R_1^{D\iota}]$ for $\iota \in \{v, s\}$ and all $i \in I$. Banks with public digital currency deposits can borrow $L_1^{CB} = (1 - \lambda)\int_{i\in I} M_1^{Cswi}$ in public digital currency from the central bank, and the bank does not default. A marginal late consumer $i' \in I$ prefers to deviate and not withdraw at $t = 1$, $w^{si'} = M_1^{Cswi'} = 0$. Withdrawing at $t = 2$ provides the late consumer a greater withdrawal return and hence a greater amount of goods bought at $t = 2$ for consumption. Hence, the run threat does not materialize, and a digital currency run equilibrium does not exist.

In equilibrium, banks do not borrow from the central bank. The potential case of a digital currency run is an out-of-equilibrium threat that is prevented from occurring as an equilibrium because of the ability and willingness of the central bank to elastically supply its digital currency as lender of last resort.

Lemma 3 *For all realizations of (λ, r_2), the Nash equilibrium without digital currency runs for banks with fiat and public digital currency deposits is unique.*

Public versus private digital currency The central bank cannot lend private digital currency to banks with private digital currency deposits that face digital currency run threats. Hence, banks cannot fend off such threats, and the digital currency run is an equilibrium. The central bank is not able to act as lender of last resort because it cannot create the private digital currency required to lend. While a private digital currency does not cause a digital currency run equilibrium to occur, the private digital currency enables it to happen.

Proposition 9 *For banks with public digital currency deposits facing a digital currency run threat, the central bank acts as lender of last resort by providing an elastic outside money supply. The digital currency run equilibrium does not exist, and the Nash equilibrium without bank runs is unique. Whereas, for banks with private digital currency deposits, the central bank cannot act as lender of last resort, and the digital currency run equilibrium exists at dates with* $\lambda < \frac{1}{1+m^{\iota}_{0,\tau}}$.

The proposition reflects the contrast of the elastic supply of public digital currency but inelastic supply of private digital currency. For a public digital currency, the central bank can elastically supply its own digital currency to banks. For a private digital currency, the central cannot create the private digital currency required for lender of last resort.

This result also highlights a distinction between an elastic value yet inelastic supply of a private digital currency. Even with an inelastic supply of the digital currency, prices are elastic and permits the optimal equilibrium, even with the realization of low asset returns and high early consumer liquidity needs. However, an inelastic supply of the digital currency also permits the digital currency run equilibrium, which elastic prices do not prevent. Hence, there is a trade-off for private digital currency deposits, which avoid the costs of distortionary central bank fiat inflation but are subject to digital currency runs.

6 Concluding remarks

A major theme in the academic literature since the financial crisis is investigating causes of fragility in the leveraged financial system. Now, with the heightened interest and concern about the potential impact on the financial system that may come from fintech, understanding the financial fragility that major financial technologies may bring is crucial.

This paper provides a first examination within the burgeoning literature on fintech of the potential impact of digital currency on the stability of the banking system. Digital currency permits but does not necessarily lead to the ex-ante disintermediation of the banking system. Consumers may deposit digital currency at banks because of the benefit of liquidity risk sharing that banks provide. Banks are resilient from

aggregate return and liquidity risk with an elastic price level under a digital currency as with fiat money. The disintermediation threat of digital currency takes the form of digital currency runs that create fragility of the banking system ex interim.

This paper shows that there is an important trade-off between the features of privately issued digital currency, such as bitcoin, and publicly issued central bank digital currency, which is a growing consideration by central banks worldwide. Central bank discretion can lead to distortionary inflation that impacts public digital currency as with fiat money but enables the central bank to act as lender of last resort. Private digital currency precludes the central bank from inflation but also from acting as a lender of last resort. The central bank can elastically supply its own digital currency, as with fiat money, to lend to banks with public digital currency deposits. This prevents digital currency runs from occurring and provides a unique equilibrium with the optimal allocation of liquidity and consumption. However, the inelastic supply of private digital currency allows for a banking crisis equilibrium with digital currency runs that deplete banks with private digital currency deposits.

Appendix: Proofs

Proof for Proposition 1. The planner's optimization (7) gives binding budget constraints and first order conditions for EU with respect to a_0, which gives equation (8); g_1, which gives the first equation of (14); and a_1, which gives the third equation of (14). Binding budget constraints imply equations (9)-(11).

The optimal storage, liquidation, and consumption allocation depends on the joint realization of (λ, r_2). Define consumption if there were no storage or liquidation for any realization of (λ, r_2), $g_1(\lambda, r_2) = a_1(\lambda, r_2) = 0$, as $\breve{c}_1 \equiv \frac{g_0^*}{\lambda}$, $\breve{c}_2 \equiv \frac{a_0^* r_2}{1-\lambda}$. For $u'(\breve{c}_1) < u'(\breve{c}_2)$, there is positive storage $g_1^* = (1-\lambda)g_0^* - \lambda a_0^* r_2 > 0$ to equalize marginal utilities between early and late consumers such that $u'(c_1^*) = u'(c_2^*)$. As a result, $c_1^* = c_2^* = g_0^* + a_0^* r_2$. This outcome occurs for a low enough joint realization of (λ, r_2), which can be expressed as $r_2 < \breve{r}_2(\lambda) \equiv \frac{(1-\lambda)g_0^*}{\lambda a_0^*}$ and $\lambda < \breve{\lambda}(r_2) \equiv \frac{g_0^*}{g_0^* + a_0^* r_2}$ that implies a threshold $(\breve{\lambda}, \breve{r}_2)$. When the illiquid asset return or the aggregate liquidity need for early consumers is small enough, positive storage of goods from $t = 1$ to $t = 2$ enables late consumers to share equally with early consumers in the total goods available at $t = 1$ and $t = 2$. The marginal rate of substitution between late and early consumers equals the marginal rate of transformation of one on storage between $t = 2$ and $t = 1$.

For $u'(\breve{c}_1) > \frac{r_2}{r_1} u'(\breve{c}_2)$, which holds with an implicit $(\hat{\lambda}, \hat{r}_2)$ for $r_2 > \hat{r}_2(\lambda)$ and $\lambda > \hat{\lambda}(r_2)$, such a high enough joint realization of (λ, r_2) implies there is instead positive liquidation $a_1^* > 0$ implicitly defined by $u'(c_1^*) = \frac{r_2}{r_1} u'(c_2^*)$. When the illiquid asset return or the aggregate liquidity for early consumers is large enough, asset liquidations allows for early consumers to share in part of the abundance of goods that are available at $t = 2$. The marginal rate of substitution between late and early consumers equals the marginal rate of transformation between assets' return at $t = 2$ and liquidation return at $t = 1$.

Otherwise, for $u'(\breve{c}_1) \in \left[u'(\breve{c}_2), \frac{r_2}{r_1} u'(\breve{c}_2)\right]$, for moderate realizations of (λ, r_2), there is no storage or liquidation, $g_1^* = a_1^* = 0$, hence $u'(c_1^*) \in \left[u'(c_2^*), \frac{r_2}{r_1} u'(c_2^*)\right]$. These results for optimal consumption, storage, and liquidation are summarized as

$$u'(c_1^*) = \begin{cases} u'(c_2^*) & \text{for low } (\lambda, r_2), \quad g_1^* > 0 \\ \in \left[u'(c_2^*), \frac{r_2}{r_1} u'(c_2^*)\right] & \text{for moderate } (\lambda, r_2), \quad g_1^* = a_1^* = 0 \\ \frac{r_2}{r_1} u'(c_2^*) & \text{for high } (\lambda, r_2), \quad a_1^* > 0. \end{cases} \tag{14}$$

Proof for Proposition 2. Market clearing for goods at $t \in \{0,1,2\}$ requires that all constraints bind for the optimizations of the consumer, bank, and firm given by optimization equations (3)-(5), with the exception of the firm's constraint $a_1 \leq a_0$. Necessary first order conditions and sufficient second order conditions hold for the consumer, bank, and firm optimization. Thus, the market equilibrium exists and is unique up to an indeterminate price level at $t = 0$, $P^s_{0,0}$, with equilibrium prices $P^s_t(\lambda, r_2)$ at $t \in \{1,2\}$ given by equations (12) and (13), and where first order conditions for the bank's optimization determine optimal deposit rates as $R^{Ds}_1 = \frac{g^*_0}{\lambda}$ and $R^{Ds}_2 = \frac{a^*_0 \bar{r}_2}{1-\lambda}$ and loan rates as $R^F_1 = 1$ and $R^{Fs^*}_2(\lambda, r_2)$.

Substituting with equilibrium prices from equations (12) and (13) into the budget constraints for the consumer, bank, and firm; applying market clearing conditions; and simplifying; there is no bank borrowing from the central bank, $L^{CB}_1(\lambda, r_2) = 0$, and the firm and bank default fractions equal one, showing no bank defaults, $\delta^D_t(\lambda, r_2) = 1$ for $t \in \{1,2\}$, for any $\beta^{CB} \leq 1$.

With $\beta^{CB} = 1$, since the central bank's objective function is equivalent to that for banks, the expected utility of consumers EU, the central bank optimally sets its rate R^{CB}_2 equal to the market equilibrium rate $R^{Fs^*}_2$ on loans to firms made at $t = 1$ that exists without consideration of the central bank optimization (6): $R^{CB^*}_2(\lambda, r_2) = R^{Fs^*}_2(\lambda, r_2)$.

Loans to firms made at $t = 1$ have a real return $r^{Fs^*}_2(\lambda, r_2) \equiv \frac{R^{Fs^*}_2(\lambda, r_2)}{\Pi^s_2(\lambda, r_2)}$. The firm's first order conditions with respect to $\{g_t, a_t\}_{t\in\{1,2\}}$ determine $a_t(\lambda, r_2) = a^*_t(\lambda, r_2)$ and $g_t(\lambda, r_2) = g^*_t(\lambda, r_2)$ for $t \in \{0,1\}$, where for $\lambda < \check{\lambda}(r_2)$, $r^{Fs}_2 = 1$; for $\lambda \in (\check{\lambda}(r_2), \hat{\lambda}(r_2))$, $r^{Fs} \in (1, \frac{r_2}{r_1})$); and for $\lambda \geq \hat{\lambda}(r_2)$, $r^{Fs} = \frac{r_2}{r_1}$. Thus, $q^s_1 = q^{s*}_1 = g^*_0 + a^*_1 r_1 - g^*_1$, and $q^s_2 = q^{s*}_2 = (a^*_0 - a^*_1)r_2 + g^*_1$. From the consumption equation set (2) and prices in equations (12) and (13), consumption for early and late consumers can be solved as

$$c_1 = \frac{D^s_0 R^{Ds}_1}{P^s_1} = \frac{q^s_1}{\lambda} \tag{15}$$

$$c^i_2 = \frac{D^s_0 R^{Ds}_2}{P^s_2} = \frac{q^s_2}{1-\lambda}, \tag{16}$$

which for $q^s_t = q^{s*}_t$ gives $c_t = c^*_t$ for $t \in \{1,2\}$.

Proof for Corollary 1. Since consumers have nominal revenues at $t = 0$ of P^s_0 from selling their one unit of goods endowment, their deposits are $D^s_0 = P^s_0$, and expected

prices are

$$\begin{aligned} E[P^s_{1,\tau}(\lambda, r_2)] &= E[\tfrac{\lambda g^*_0}{\bar{\lambda} c^*_1} P^s_{0,\tau}] = E[P^s_{0,\tau}] \\ E[P^s_{2,\tau}(\lambda, r_2)] &= E[\tfrac{(1-\lambda) a^*_0 \bar{r}_2}{(1-\bar{\lambda}) c^*_2} P^s_{0,\tau}] = E[P^s_{0,\tau}]. \end{aligned}$$

Since the period $t = 0$ price level at date $\tau = 0$ is normalized to one, $P^s_{0,0} = 1$, $E[P^s_{t,0}] = 1$. Since $P^s_{0,\tau} = P^s_{2,\tau-1}$, we have $E[P^s_{0,1}] = 1$, and hence $E[P^s_{t,\tau}] = 1$, which implies $E[\Pi^s_{t,\tau}] = 1$ and $E[\frac{P^s_{t,\tau+1}}{P^s_{t,\tau}}]$ for $t \in \{0, 1, 2\}$.

Proof for Proposition 3. From the central bank's optimization (6), the first order condition with respect to $R^{CB}_2(\lambda, r_2)$ implies that $\hat{R}^{CB}_2(\lambda, r_2) > R^{Fs^*}_2(\lambda, r_2)$. The bank's first order conditions with respect to L^{Fs}_1 and L^{CB}_1 require $R^{Fs}_2 = R^{CB}_2$, hence $R^{Fs}_2(\lambda, r_2) > R^{Fs^*}_2(\lambda, r_2)$ and $r^{Fs}_2(\lambda, r_2) > r^{Fs^*}_2(\lambda, r_2)$.

If $\beta^{CB} < 1$ is unexpected, then $a_0 = a^*_0$ and $g_0 = g^*_0$ are unchanged. The firm's first order conditions imply that $g_1 \leq g^*_1$ and $a_1 \geq a^*_1$, with $\hat{q}^s_1 > q^{s*}_1$ and $\hat{q}^s_2 < q^{s*}_2$. If $\beta^{CB} < 1$ is expected, the firm's first order conditions imply that $a_0 < a^*_0$ and $g_0 > g^*_0$, which implies that $\hat{q}^s_1 > q^{s*}_1$ and $\hat{q}^s_2 < q^{s*}_2$. Hence, in either case, $\hat{c}^s_1 > c^{s*}_1$, $\hat{c}^s_2 < c^{s*}_2$, and $\hat{\Pi}^s_2 > \Pi^{s^*}_2$.

Proof for Lemma 1. With public digital currency, equilibrium prices at $t \in \{1, 2\}$ are

$$\begin{aligned} P^s_1(\lambda, r_2) &= \frac{\lambda(D^s_0 R^{Ds}_1 + M^{Cs}_0) + (1-\lambda)\int_{i \in I}(M^{Cs}_0 - M^{Csi}_1}{q^s_1} \\ P^s_2(\lambda, r_2) &= \frac{(1-\lambda)\int_{i \in I}(D^s_0 R^{Ds}_2 + M^{Csi}_1)}{q^s_2}. \end{aligned}$$

Hence, $\frac{R^{Ds}_t}{P^s_t} > \frac{1}{P^s_t}$ for $t \in \{1, 2\}$, which implies from the consumer's first order conditions that $M^{Cs}_0 = M^{Csi}_1 = 0$.

Proof for Lemma 2. Inflation at $t = 2$ of any date τ, $\Pi^s_{2,\tau}(\lambda, r_2) = \frac{P^s_2}{P^s_1}$, is independent of $D^s_{0,\tau}(M^s)$. Hence, the firm's real return $r^{Fs}_2(\lambda, r_2) \equiv \frac{R^{Fs}_2(\lambda, r_2)}{\Pi^s_2(\lambda, r_2)}$ is independent of $D^s_{0,\tau}(M^s)$, which implies that for $t \in \{1, 2\}$, q^s_t, and thus c_t given by equations (15) and (16), are independent of $D^s_{0,\tau}(M^s)$ and M^s.

Proof for Proposition 4. Note that if private digital currency is held outside of the banking system, market clearing for private digital currency for $\iota = v$ at periods

$t \in \{0,1,2\}$ can be written as:

$$\begin{aligned} t &= 0\text{: } M_{0,\tau}^{C\iota} = M_{2,\tau-1}^{F\iota} + \mathbf{1}_{\tau=\tau'} M^{\iota} - M_{0,\tau}^{F\iota} \\ t &= 1\text{: } M_{1,\tau}^{F\iota} = \lambda M_{0,\tau}^{C\iota} + (1-\lambda)\textstyle\int_{i\in I}(M_{0,\tau}^{C\iota} - M_1^{C\iota i}) + M_{0,\tau}^{F\iota} \\ t &= 2\text{: } M_{2,\tau}^{F\iota} = (1-\lambda)\textstyle\int_{i\in I} M_{1,\tau}^{C\iota i} + M_{1,\tau}^{F\iota} = M_{2,\tau-1}^{F\iota} + \mathbf{1}_{\tau=\tau'} M^{\iota}. \end{aligned}$$

The private digital currency prices for goods at periods $t = 1$ and $t = 2$ are

$$\begin{aligned} P_{1,\tau'}^{v} &= \frac{\lambda M_0^{Cv} + (1-\lambda)\int_{i\in I}(M_0^{Cv} - M_1^{Cvi})}{q_1^v} \\ P_{2,\tau'}^{v} &= \frac{(1-\lambda)\int_{i\in I} M_1^{Cvi}}{q_2^v}, \end{aligned}$$

respectively. The firm's first order conditions with respect to $\{M_{t,\tau}^{Fv}, q_t^v\}_{t\in\{0,1,2\}}$ and market clearing for private digital currency and goods at periods $t \in \{0,1,2\}$ requires that $E_{0,\tau}[\frac{P_{0,\tau+1}^v}{P_{2,\tau}^v}] = 1$, $E_{0,\tau}[\frac{P_{1,\tau+1}^v}{P_{2,\tau}^v}] = 1$, $E_{0,\tau}[\frac{P_{2,\tau+1}^v}{P_{2,\tau}^v}] = \frac{1}{\bar{r}_2}$, and $\frac{P_{1,\tau}^v}{P_{2,\tau}^v} = \frac{q_{2,\tau}^v}{q_{1,\tau}^v}$, which implies that $E_{0,\tau}[c_1] = E_{0,\tau}[c_1^i] = E_{0,\tau}[\frac{1}{P_{1,\tau}^v}] = 1$, $E_{0,\tau}[c_2^i] = E_{0,\tau}[\frac{1}{P_{2,\tau}^v}] = \bar{r}_2$, and that the consumption for late consumers buying goods at $t = 2$ is weakly greater than at $t = 1$, $c_2^i \geq c_1$; hence, late consumers store all private digital currency at $t = 1$: $M_1^{Cvi} = M_0^{Cv}$.

Proof for Corollary 2. The result follows directly from $E_{0,\tau}[\frac{P_{0,\tau+1}^v}{P_{2,\tau}^v}] = 1$, $E_{0,\tau}[\frac{P_{1,\tau+1}^v}{P_{2,\tau}^v}] = 1$, and $E_{0,\tau}[\frac{P_{2,\tau+1}^v}{P_{2,\tau}^v}] = \frac{1}{\bar{r}_2}$ in the proof of proposition 4.

Proof for Proposition 5. Following the proof of proposition 2 the market equilibrium provides the optimal first best consumption c_1^* and c_2^* with no bank defaults, $\delta_1^D = \delta_2^D = 1$, for all realizations of (λ, r_2), for private digital currency held as bank deposits equivalently to the case of fiat money deposits for $\beta^{CB} = 1$. In particular, equilibrium prices for private digital currency are

$$P_1^v(\lambda, r_2) = \frac{\lambda(D_0^v R_1^{Dv} + M_0^{Cv})}{q_1^s} \tag{17}$$

$$P_2^v(\lambda, r_2) = \frac{(1-\lambda)(D_0^v R_2^{Dv} + M_0^{Cs})}{q_2^v}, \tag{18}$$

with $R_t^{Dv} = R_t^{Ds}$ and $R_t^{Fv} = R_t^{Fs}$ for $t \in \{1,2\}$. Hence, $\frac{R_t^{Ds}}{P_t^s} > \frac{1}{P_t^s}$ for $t \in \{1,2\}$, which implies that consumers prefer to hold private digital currency in bank deposits rather than directly: $M_0^{Cs} = M_1^{Csi} = 0$.Thus, consumption for early and late consumers

holding private digital currency deposits is

$$c_1 = \frac{D_0^v R_1^{Dv}}{P_1^v} = \frac{q_1^v}{\lambda} \tag{19}$$

$$c_2^i = \frac{D_0^v R_2^{Dv}}{P_2^v} = \frac{q_2^v}{1-\lambda}, \tag{20}$$

which for $q_t^v = q_t^{s*}$ gives $c_1 = c_1^*$ and $c_2 = c_2^*$ for $t \in \{1, 2\}$.

Proof for Corollary 3. This result follows directly from the results and proofs from propositions 1 and 5.

Proof for Proposition 6. From proposition 3, for $\beta^{CB} < 1$, the expected utility of fiat money deposits and public digital currency is less than that of the optimal consumption allocation $\{c_t^*\}_{t\in\{1,2\}}$, From the proof of proposition 5, regardless of β^{CB}, the expected utility of private digital currency deposits is equal to that of the optimal consumption allocation $\{c_t^*\}_{t\in\{1,2\}}$. Hence, consumers do not hold fiat money deposits or public digital currency, $D_0^s = M_0^{Cs} = 0$, and only hold private digital currency deposits D_0^v.

Proof for Proposition 7. Consider a withdrawal strategy set σ without early withdrawals, $w^{\iota i}(\lambda, r_2) = 0$, which for feasibility requires $M_1^{C\iota wi}(\lambda, r_2) = 0$, for all $\lambda \in (0, 1)$, $r_2 \in (0, r_2^{\max})$, and $i \in I$. Consumption for depositors is equivalent to that from propositions 2, 3 and 5, with optimal consumption for fiat money deposits with $\beta^{CB} = 1$ and for private digital currency deposits, and with suboptimal consumption for fiat money deposits with $\beta^{CB} < 1$.

In particular, a late consumer's consumption at $t = 2$ is $c_2^i = \frac{D_0^\iota R_2^{D\iota}}{P_2^\iota}$ for $\iota \in \{v, s\}$. Suppose there is a deviation withdrawal strategy $\sigma^{i''}$ by a late consumer i'', such that $w^{\iota i''}(\lambda, r_2) > 0$ and $M_1^{i''}(\lambda, r_2) \leq w^{\iota i''}(\lambda, r_2)$ for any $\lambda \in (0, 1)$ and $r_2 \in (0, r_2^{\max})$. This late consumer's consumption is $c_1^{i''} + c_2^{i''}$, where $c_1^{i''} = \sum_\iota \frac{w^{\iota i''} D_0^\iota R_1^{D\iota} + M_0^{C\iota} - M_1^{C\iota i} - M_1^{C\iota wi}}{P_1^\iota}$, $c_2^{i''} = \frac{(1-w^{\iota i''}) D_0^\iota R_2^{D\iota} + M_1^{C\iota i} - M_1^{C\iota wi}}{P_2}$, and hence $c_1^{i''} + c_2^{i''} < c_2^i$. Thus, given the withdrawal strategy set σ, including the withdrawal strategies for late consumers $i' \neq i$, $\{\sigma^{i'}\}_{i'\in I}$, where $\sigma^{i'} = \{0, 0\}$; $\sigma^i = \{0, 0\}$ is a weakly best response for all (λ, r_2) and a strictly best response for $\lambda(r_2) > \check{\lambda}(r_2)$. Hence, σ is a Nash equilibrium of the withdrawal game.

Proof for Proposition 8. Consider a withdrawal strategy set σ with complete early withdrawals, $w^{\iota i} = 1$ in the form of demands for digital currency, $M^{C\iota wi} = D_0^\iota R_1^{D\iota}$, for

all late consumers $i \in I$. For the case of $M_0^{B\iota} < D_0^\iota R_2^{D\iota}$, it is not feasible to pay these early withdrawal demands in currency, which implies the bank defaults at $t = 1$. For the case of $(1-\lambda)D_0^\iota R_1^{D\iota} > M_0^{B\iota}$, the bank's budget constraint at $t = 1$ implies that the bank defaults at $t = 1$, $\delta_1^D < 1$, does not roll over any lending to firms, $L_1^{F\iota} = 0$, and hence has no revenues for withdrawals at $t = 2$, for a complete default at $t = 2$, $\delta_2^D = 0$.

Suppose there is any deviation in the withdrawal strategy $\sigma^{i''}$ by any late consumer i''. For $w^{\iota i''} < 1$, .the late consumer receives no amount for the withdrawal of $(1-w^{\iota i''})$ at $t = 2$. For an early withdrawal demand not in digital currency, consumption $c_1^{i''}+c_2^{i''}$ is unchanged. Hence, σ is a Nash equilibrium.

Proof for Lemma 3. Consider any withdrawal strategy set σ with positive early withdrawals for any set $I' \in I$ of late consumers. The bank can borrow from the central bank the amount of the public digital currency withdrawals at $t = 1$: $L_1^{CB} = (1-\lambda)\int_{i' \in I'} M_1^{Cswi'}$. There is no default for the bank, which implies that the withdrawal strategy with a positive amount of early withdrawals for each late consumer $i' \in I'$ is not a best response. Hence, the Nash equilibrium without early withdrawals is unique.

Proof for Proposition 9. Following from the proof of lemma 3, for banks with public digital currency deposits, there is a unique Nash equilibrium without early withdrawals by late consumers. For banks with private digital currency deposits, for $\lambda < \frac{1}{1+m_{0,\tau}^\iota}$, consider complete withdrawals in the form of digital currency by all late consumers. For any positive amount of bank borrowing in the form of fiat money from the central bank, $L_1^{CB} > 0$, the bank would default on repaying the central bank at $t = 2$, $\delta_2^{CB} < 1$, which rules out such borrowing in equilibrium: $L_1^{CB} = 0$. Hence, following the proof of proposition 8, the withdrawal run is a Nash equilibrium.

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